ASSESSING THE EFFECTS OF TEACHING ON THREE TENTH-GRADE STUDENTS' CONCEPT IMAGES OF QUADRATIC FUNCTIONS

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Abstract.
This paper explores the effects of teaching on the concept images, with respect to quadratic functions, of three tenth-grade students – one high-performing, one medium-performing, and one low-performing. Pre- and post-teaching interview data, focusing on the students' verbal, symbolic, tabular and graphical representations of quadratic functions, and on their affective responses to those representations, are summarized. At the post-teaching stage, after a sequence of 10 lessons on elementary algebraic functions, the high-performing interviewee enjoyed grappling with quadratic function tasks. During lessons he incorporated symbolic representations of quadratic functions into his thinking, and his mode of reasoning began to change from inductive to deductive. The medium performing interviewee liked solving only those quadratic function questions that she thought she could answer correctly. Her concept image continued to rely almost exclusively on reading coordinates from a given graph and was not connected to more symbolic ways of thinking about quadratic functions. The low-performing interviewee did not like questions concerning quadratic functions, and did not think he could answer them correctly. Nevertheless, his concept image with respect to quadratic functions began to feature more accurate and richer cognitive connections.

Keywords: Concept image, Representation, Quadratic function, Interview, Ten-grade student

The Aim and Design of the Study, and Definitions of Related Terminology

The study described in this paper explored effects of 10 lessons concerned with linear and quadratic functions on the concept images with respect to quadratic functions of three tenth-grade students – one high-performing, one medium-performing, and one low-performing. The criterion for deciding whether a student was high-, medium-, or low-performing was performance on a pencil-and-paper pre-teaching *Function Test* designed to measure students' ability to represent algebraic functions in different ways – especially graphically, verbally, symbolically, and tabular – and then to be able to connect different representations so that, in a sense, a coordinated, *reified* form of the concept of a quadratic function was achieved (Sfard, 1991).

By the term "concept image" is meant "the cognitive structures in an individual's mind that are associated with a given concept." These include "all the mental pictures and associated properties and processes" that exist as a result of the individual's experience with examples and non-examples of the concept (Tall & Vinner, 1981, pp. 151-152). According to Vinner and Hershkowitz (1980), in most cases students call to mind the concept image, and not the formal, verbal, concept definition, when dealing with a concept. If a student's concept image conflicts with the formal definition accepted by the mathematical community, he/she may consider the formal theory to be "inoperative and superfluous" (Tall & Vinner, 1981, p. 154). Thus, if we can identify a student's concept image for a mathematical concept, then we will be better placed to understand that student's thinking, and, if necessary, to assist the student to correct misconceptions.

In the study described here the concept images with respect to quadratic functions of three Grade 10 students in Chiang Mai, Thailand, are investigated. Two principal forms of data were analyzed: (a) student responses to pre- and post-teaching written questions designed to illuminate the students' verbal, symbolic, tabular and graphical representations of quadratic functions; and (b) pre- and post-teaching data gained from interviewing the students before and after the 10 lessons. Data relating to the students' affective responses to quadratic functions and their forms of representations are also briefly summarized.

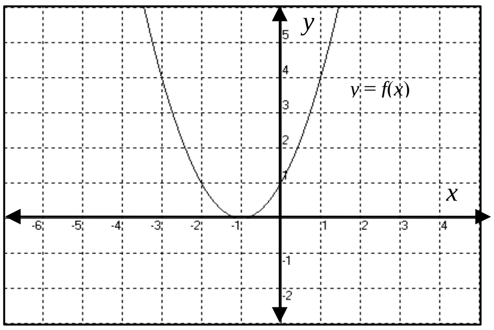
The Semi-Structured Interview Protocol Used in the Study

In previous studies (Vaiyavutjamai & Clements, 2006a, 2006b) the author used an interview protocol that combined interview approaches developed by Newman (1983) and Booth (1984), and this was once again adopted. With Newman's (1983) diagnostic interview technique the interviewer makes five key requests of interviewees: (a) Please read the question to me; (b) Tell me, what does the question mean? (c) What will you need to do to answer this question? (d) Now answer it, and tell me what you are thinking as you answer it. (e) Now write down your final answer. Newman classified errors made in response to these five requests as Reading, Comprehension, Transformation, Process Skills, and Encoding errors, respectively.

Booth (1984) used an interview schedule that was essentially an extension of the Newman approach. Booth's semi-structured approach included requests equivalent to all five Newman requests and, in addition, she made the following five requests aimed at finding out whether interviewees: (a) knew what their answers meant in relation to the original question; (b) could check their answers; (c) would stick to their answers if challenged with other possibilities; (d) could identify other questions similar to a question they had just answered; and (e) could generalize questions to solve more complex, but nonetheless similar, tasks.

In the present study, the interviewer used Newman's five requests and some or all of Booth's "extension" requests. The present writer conducted all the interviews. I adopted a flexible, open, technique, feeling free to ask non-standard questions during interviews. Each interview, then, was semi-structured, rather than fully structured. The mathematical content dealt with in each of the interviews was associated with four pencil-and-paper questions, all taken from the pre-teaching pencil and-paper instrument, which focused on quadratic functions. Before being interviewed each interviewee had already attempted to answer the interview questions on two occasions. The first was when the pencil-and-paper instrument had been administered to the whole class. The second was immediately before an interviewee participated in a formal interview. Each interview was conducted on a one-one basis, and the content and forms of the questions asked by the interviewer were influenced by what the student had written during the two earlier attempts at the questions.

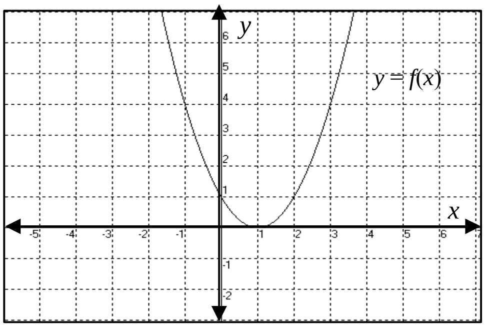
In the pre-teaching interviews, each interviewee was shown Figure 1, and in the post-teaching interview each was shown Figure 2.



1.1 Write a rule for the function *f*, in ***symbols***.
1.2. Use ***words*** to describe the rule for the function *f*.
1.3 Complete the following table of values for *f*(*x*).
The first one has already been done for you.

x	*f*(*x*)
-3	4
-2	
-1	
0	
1	

Figure 1: Graph for the first *pre*-teaching interview question, and three associated sub-tasks.



1.1. Write a rule for the function *f*, in ***symbols***.
1.2. Use ***words*** to describe the rule for the function *f*.
1.3 Complete the following table of values for *f*(*x*).
The first one has already been done for you.

x	*f*(*x*)
-1	4
0	
1	
2	
3	

Figure 2: Graph for the first *post*-teaching interview question, and three associated sub-tasks.

On each occasion the interviewee was expected to respond verbally to interview questions. All interviews were audio taped. Each of the three interviewees was interviewed on two occasions: (a) immediately *before* the sequence of 10 lessons, but after the pencil-and-paper test had been administered to the class as a pre-teaching instrument; and (b) immediately *after* the class had participated in the 10 lessons and had completed answers to a post-teaching pencil-and-paper test that was a parallel form of the pre-teaching pencil-and-paper test. An interview will be referred to as "pre-teaching" or "post-teaching," depending on when it took place with respect to the set of 10 class lessons on functions.

The Interviewees and the Teacher

Brief comments on the three interviewees and on their mathematics teacher are now provided.

Interviewee 1

Interviewee 1 was a 15-year-old boy. On the pencil-and-paper *Pre-Teaching Functions Test* he had answered 9 questions out of a possible 24, thereby gaining the highest pre-teaching score in the class. On the pencil-and-paper parallel *Post-Teaching Functions Test* he answered 22 of the 24 questions correctly. Interviewee 1 stated, during the pre-teaching interview, that he liked tackling questions related to quadratic functions even if he found them difficult. At the pre-teaching stage he tended to have *appropriate* confidence, knowing when he was correct and when he was not.

Interviewee 2

Interviewee 2 was a 15-year-old girl who was in the same middle-stream, tenth-grade mathematics class as Interviewee 1. On the *Pre-Teaching Functions Test* she gave only two correct answers out of a possible 24, but such was her class's overall low performance on the test that she was regarded as a medium- performing student for her class. On the *Post-Teaching Functions Test* she gave eight correct answers out of a possible 24. During the pre-teaching interview, Interviewee 2 stated that the only quadratic function questions that she enjoyed tackling were those that she knew she could answer correctly.

Interviewee 3

Interviewee 3, a 15-year-old boy, was in the same middle-stream, tenth-grade mathematics class as Interviewees 1 and 2. On the *Pre-Functions Test* he was the lowest performing student for his class. He did not answer any of the 24 questions correctly. On the *Post-Functions Test,* however, he gave 12 correct answers out of a possible 24. During the pre-teaching interview, Interviewee 3 stated that although he recalled studying something about quadratic functions in Grade 9, he did not remember anything about what he had been taught, then, and he did not enjoy working on quadratic equations questions now.

The Teacher

The teacher of the mathematics class that included the three interviewees was an experienced and mathematically well qualified teacher. Her main emphasis when teaching quadratic functions was on linking symbolic algebraic representations of functions with the parabolas that were the graphical representations of the functions. She believed it was important for students to recognize that the canonical form $y = a(x - h)^2 + k$ (where a, h and k can be any real numbers, except $a \neq 0$), has a parabolic graph with turning point (h, k) and orientation that depends on the sign of a.

Excerpt of a Pre-and Post-Teaching Interview Transcript Showing Effects of Teaching

Table 1 shows the English translation of excerpts from interviews with Interviewee 1. Table 2 and 3 shows some part of excerpts from interviews with Interviewee 2 and Interviewee 3.

Table 1

English Translation of Pre- and Post-Interview Transcript for a Tenth-grade High-Performing Student

Interview Question/ Request	**Excerpt of Transcript from Pre-Teaching Interview**	**Excerpt of Transcript from Post-Teaching Interview**
1. Please read the question to me.	I: Please read Question 1. S: [*He read correctly*]	I: Please read the question to me. S: [*He read correctly*]
2. Please tell me what came into your mind when you were reading the question.	I: Please tell me what came into your mind as you read the question. S: [*quiet for about 3 seconds*] I think I need to find the symbol. I: What is a symbol? S: [*Quiet for about 2 seconds*] It is an equation. I: Do you know the equation? S: No, I don't. I: You don't know the equation? S: I have read very little about equations. I: Why do you think it is an equation? S: It looks like a parabola.	I: Please tell me what came into your mind as you read the question. S: This is a quadratic function. I: It is a quadratic function. Anything else. S: [*Quiet*]
3. Do you think you understand the meaning of the question?	I: Do you understand this question? S: Write a symbol and a word from the graph. I: You have to write a symbol and a word. S: Complete the values in table. I: Do you know what the word "symbol" means? S: A number is a symbol. I: What are symbols? S: Multiply, square, number. I: Anything else? S: [*Quiet* for about 3 seconds] I: That's all? What does "word" mean? S: Translate the symbol into a word. I: What do you need to do to complete the table? S: Write numbers. I: You insert some numbers in the table.	I: Do you understand this question? S: Yes. I: What do you understand about it? S: You need to find the rule of function *f*. I: In what form will you give your answer? S: In symbolic form. I: What are symbols? S: Symbols are variables and numbers. I: What does "word "mean? S: Translate symbols to words. I: What will you give as your answer in the table? S: You substitute numbers from the domain of the function and then find numbers in the range of the function.
4. As you were reading the question, did you think of any pictures or diagrams, or of some incident that has happened to you in your life?	I: As you were reading the question, did you think of any pictures or diagrams, or of some incident that has happened to you in your life? S: [*Quiet for about 5 seconds*] I: Did you think of any pictures or diagrams, or of some incident that has happened to you in your life? .... After you read the question, did you remember anything that you could link to it? S: A parabola I: Why did you think of a parabola? S: Because the graph is a curved line. I: It is a curved line, what else did you think of that linked to the question? S: Solving equations. I: Solving equations. What else? S: [*Quiet for about 3 seconds*]	I: As you were reading the question, did you think of any pictures or diagrams, or of some incident that has happened to you in your life? S: The form of the equation. I: What do you mean by "form of the equation? S: Square of *x* minus *h*, and plus *k*. I: What kind of equation is it? S: It is a quadratic equation. I: When did you learn about quadratic equations S; I learnt from my teacher [*He named his mathematics teacher*.] I: What else came into your mind? S: Parabola, equation.
5. Do you like doing questions like	I: So, that's what you were thinking about. Do you like doing questions like this one? S: Yes, I like it. I enjoy solving the problems.	I: Do you like doing questions like this one? S: I like it I: Why do you like it?"

this one? Why?	I; Do you have any other reasons for liking it? S: It is self-evaluation. I: You evaluate yourself … whether you can solve the problem or not.	S: I can solve it.
6. Did you think you could solve this problem?	I: Do you think you can solve the problem? S: Yes … but I'm not really confident that I will do it correctly. I: What "percentage confident" are you? S: Fifty per cent. I: You think you can do part of question. The question has three sub-questions, which sub-question would you do first? S: The one on symbolic form. I: You will do the part on symbolic form – that is, Question 1.1. What will you do next? S: Question 1.3.	I: Do you think you can solve the problem? S: Yes I: Do you think you can do all of the problems correctly, 100 percent of them? S: Yes. I: What question will you do first? S: Question 1.1 I: Why? S: Because when I answer in equation form then I can write it in word form.
7. Please show me your work.	I: Please show me your work. You can "think aloud" as you do it. Write what you think on the sheet, as you do it. S: [*Quiet for almost two minutes*] S: What question will I do first? I: You make decision by yourself. What question will you do first? S: Question Three I: So you've changed your mind and you'll solve Question Three first. S: [*Wrote numbers as answers in the table.*]	I: Please show me your work. S: [*He wrote the following on the work sheet*] $(x\text{-}1)^2 + 1$ $(x\text{-}1)^2$ $(x\text{-}1)^2 + k$ (h , k) I: Have you finished Question 1.1? S: No. [*Wrote* $(x-1)^2$]
8. Please tell me what you are thinking as you answer it.	I: Please explain how you got the numbers that you wrote in the table? S: I get them from the graph. I: How did you get the numbers from the graph? S: From the question, *x* equals -3 and *y* equals 4 I: From the question, *x* equals -3 and *y* equals 4 here [*Pointed to "4" in the table*], so you gave other numbers in the table. S: *x* equals -2, then *y* equals 1. I: Please point where that is on the graph. S: *x* equals -2 and *y* equals 1 is here [*pointed at (-2, 1), correctly*]. I; Where is *x* equals -1 and *y* equals 0? S: *x* equals -1 and *y* equals 0 is here [*pointed at (-1, 0) correctly*]. I; That is your answer, -1 and 0. S: *x* equals 1, *y* equals 4 is here [*pointed at (1, 4) correctly*]. I: What question will you do next? S: [*Quiet for about 1minute* 20 seconds] I: Could you do Question One – where you have to write in symbolic form? You can guess the answer. Did you give an answer when you did the test? S: No, I didn't. I: Didn't you answer Question One? S: I only completed the table. I: You only completed the table. Can you give an answer in symbolic form?	I: Please explain why you wrote the first and second lines. S: That is incorrect. [*Pointed to the first line that he had written.*]. I: Please explain how you got the answer. S: In the beginning, I wrote it correctly, but when I substituted a number for *x*, I did it incorrectly. I: How did you know to substitute a number for *x*, and why do you know that you were incorrect? S: I got it from the minimum point… The equation is *x* minus *h*, so the minimum point is (*h, k*). I: What did you do next? S: *k* is value of *y*, so at the minimum point, *y* is zero. … *h* is *x*, which equals 1. I: How do you write it? S: Square of *x* minus 1. I: Please do Question 1.2 S: [*He wrote "Square of minus 1 from x ."*] I: Please read your answer. S: Square of minus 1 from *x*. I; Please do Question 1.3. S; Substitute for *x* the numbers that were given. I: Where did you get the numbers to substitute for *x*? S: I substituted numbers that I got from

	S: I'd have to write it down. I: You have to write it. Can you do this question? S: [*Quiet for 5 seconds. It seemed that he did not know how to do Question 1.1*] I: Never mind. You can't do it. I understand because you haven't learnt it in class, yet. It was good that you could answer a part of the question.	Question 1.1. I: Please complete the table. S: **[***Wrote numbers as answers in the table***]** I: Pleases explain how you got the answers. S: x equals 0, so I replace x by 0, 0 minus 1 equals -1, and -1 squared equals 1.If x equals 1, 1 minus 1 equals 0 and 0 squared equals 0. 2 minus 1 equals 1 and 1 squared equals 1. 3 minus 1 equals 2 and 2 squared equals 4.
9. Now write down your final answer.	I; Please give me your final answer. S: [*He read the answer from his table*]	I; Please give me your final answer. S: [He *read the answer that he had written*]

Table 2

English Translation of Pre- and Post-Interview Transcript for a Tenth-grade Medium-Performing Student

Interview Question/ Request	**Excerpt of Transcript from Pre-Teaching Interview**	**Excerpt of Transcript from Post-Teaching Interview**
6. Did you think you could solve this problem?	I: Do you think you can solve the problem? S: No. I: Can you do any part of the problem? S: Question 1.3	I: Do you think you can solve the problem? S: I can do only Question.3.
7. Please show your work.	I: You can do Question 1.3. Please show your work. S: [*Worked for one minute, during which time she calculated the correct answers and entered them into the Table.*]	I: Please do Question 1.3. As you do it, try to write what you are thinking on the sheet. S: Do I have to show all my work? I: As much as you can. Make sure you give me your answers in table. S: [*Wrote numbers as answers in the table*]
8. Please tell me what you are thinking as you answer it.	I: Please explain how you got your answers. S: I don't know how to explain it. I: You can point to the graph. S: [*Pointed at the point with coordinates* (-3, 4)] I: Why did you indicate the point (-3, 4)? S: I look at the intersection of lines. I: Which line did the -3 come from? S: I got the x number. I; How about the 4? S: I got the y number. I; How about others numbers? S: When the x number is -2, the y number is 1. When the x-number is -1, the y-number is 0. When x is 0, y is 1. When x is 1, y is 4. I: Can you do the other questions? S: [*She tried, for about 10 seconds, and then said* …] I don't know how to do them.	I: Please explain how you get the answer. S: I find intersection point by going from the x-axis and the y-axis I: Please show me what you mean. S: [She indicated the points with coordinates (-1, 4), (0, 1) (1, 0), (2, 1) and (3, 4) and explained how substituting -1 for x meant you got 4 for y. She gave similar explanations for (0, 1), (1, 0), (2, 1) and (3, 4). I: Could you try to do the other questions? S: I don't think that I can do them. I: Could you guess the answers? S: [*Quiet*] I: OK. …….

Table 3

English Translation of Pre- and Post-Interview Transcripts for a Tenth-grade Low-Performing Student

Interview Question/ Request	Excerpt of Transcript from Pre-Teaching Interview	Excerpt of Transcript from Post-Teaching Interview
6. Did you think you could solve this problem?	I: Do you think you can solve the problem? S: No, I can't do. I: Will you try? S: Yes, I will try. I: What question will you do first? S: I don't know how to do any of them. I: You can't do any of the questions? S: I have never learnt it before. I: Never mind. You can't do the questions.	I: Do you think you can solve the problem? S: I will try. I: Are you confident you can answer the question correctly? S: I am not really confident. I: How confident are you? S: I've got a 50-50 chance of being correct. I: Which question will you answer first? S: Question 1.3
7. Please show your work.		I: Please show your work. S: [*Wrote numbers as answers in the table, for about 1.5 minutes.*]
8. Please tell me what you are thinking as you answer it.		I: Please explain how you got your answers. S: I got the answers from the graph. I: How did you get the answers from the graph? S: When x equals 0, y equals 1. [*Indicated the point with coordinates* (0, 1).] I: Please show how you got the other answers. S: [*Indicated the points with coordinate (0, 1), (1, 0), (2, 1), and (3, 4) and explained why these gave correct answers.*] ……. S: [*He tried to do Question 1.1 and became confused when trying to write his answer. Finally he wrote his answer to Question 1.1 as* $\{y/y = 0 \leq y \leq 4\}$].

Some Tentative Conclusions

Analysis of the interview excerpts in Table1 suggested that the concept images, with respect to quadratic functions, of Interviewee 1 changed qualitatively between the pre- and post-teaching stages. In line with his teacher's emphasis on deductive, rather than inductive, thinking, Interviewee 1 was beginning to learn to link the position and orientation of the graph of a quadratic function with the canonical form $a(x - h)^2 + k$. Space considerations prevented relevant but lengthy pre- and post-teaching interview excerpts being shown for Interviewees 2 and 3. However, analyses of excerpts shown in Table 2 and Table 3 indicated that these students had not followed the teaching approaches in the sequence of 10 lessons, and in particular had not learned to link the canonical abstract form with the location and orientation of the associated parabolas. Both had not moved beyond inductive identification of coordinates on a given graph. It seemed that Interviewee 2 had become thoroughly confused by the abstract, deductive approach of her teacher, and her concept image remained uncoordinated and dominated by misconceptions. Interviewee 3, on the other hand, had developed some relevant inductive approaches that he could apply accurately. However, he had not learned to link the various forms of representation and to think deductively.

At the pre-teaching stage, Interviewee 1's knowledge of the language of functions was extremely basic. Although he used the word "parabola" appropriately, and he knew how to identify the x- and y-coordinates of points on a given Cartesian plane, he had no well-established imagery or concept definitions with respect to functions, in general, and to quadratic functions, in particular. By contrast, at the post-teaching stage, his concept image with respect to quadratic functions had developed, but in a very formal way. He seemed to think that a parabola with a turning point (h, k) had to have an equation of the form $y = (x - h)^2 + k$, and he applied this idea to get the equation $y = (x - 1)^2$ for the parabola shown in Figure 2 (which had a minimum turning at (1, 0)). However, there was no hint in the interview transcript that he was aware that a given parabola on a Cartesian plane might have an equation of the form $y = a(x - h)^2 + k$, with $a \neq 1$. Nor was there strong evidence that a parabola with equation $y = (x - h)^2 + k$ was linked, in his mind, to the parabola with equation $y = x^2$. Overall, even after having participated in the 10-lesson sequence on linear and quadratic functions, his concept image still seemed to be largely inductive. He was not yet able to apply deductive thinking accurately in relation to quadratic functions because of his lack of understanding of how symbolic, verbal, graphical and tabular representations could be seen as different aspects of the same, quadratic function.

Nevertheless, at the post-teaching stage, Interviewee 1's concept image had developed, and was still developing, in what his teacher regarded as the right direction. That was *not* the case, however, for the concept images of Interviewee 2 and Interviewee 3. Although both of these students had participated in the same 10 lessons as Interviewee 1, they did not choose to apply the canonical form approach to the questions shown in Figure 2.

The results of the analyses presented in this paper are parallel to results I have found in previous studies (e.g., Vaiyavutjamai, 2009) – specifically, concept images that many secondary school students develop with respect to quadratic functions are often not consistent with the instruction they receive in class. The implication is that teachers need to take into account the wide range of concept images held by the students in their classes. Research on how that can best be achieved is urgently needed.

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