The Use of Novel Items to Uncover Lapses in Mathematical Reasoning

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Abstract

Reasoning is integral to mathematics. However it is often difficult, in a written assessment, to judge if a student had possessed the correct mathematical reasoning behind a page of right answers. This paper reports the use of novel items and interviews to probe into the mathematical thinking behind the mathematical solutions of Grade 12 (pre-university) students in Singapore in an attempt to gain an insight into the lapses in mathematical reasoning. Of particular interest in this study was an examination into what students accepted as valid mathematical explanations and justifications and how they justified their mathematical arguments. Some novel items were developed and administered to a group of 69 Grade 12 students and the errors recorded. Further information on the students' reasoning processes was obtained through semi-structured interviews of a selected group of students from the sample. The responses to the items and the interviews were analyzed in the light of existing literature. The key finding was that many mathematical errors arose because students had overly relied on instrumental understanding, which limited their transfer of knowledge to new environments. Another important finding was that inductive strategies from previous experiences in deriving mathematical generalizations were mistakenly accepted as valid mathematical proofs.

Introduction

The assessment cycle begins with the elicitation of information of a student's understanding on the subject. The information is then interpreted and used as evidence of the student's attainment of learning outcomes to decide on further actions. It is only when assessors have gathered accurate information that appropriate actions in line with the intended purpose could be taken. In a pen and paper test, it is often assumed that a student has the correct understanding if he has written the correct answer. However, this is often not the case; especially when objective of the assessment is to elicit information on the reasoning processes in problem solving contexts.

This study highlights some possible lapses in mathematical reasoning, some of which would have been overlooked if we had simply examined the correctness in the presentation. The specific research questions include: What are some of the mathematical reasoning lapses made by students? What may be the related causes of these lapses? It is expected that answers to these questions will provide teachers with insights into the reasoning lapses behind students' mistakes. This could help teachers anticipate students' errors and plan assessment tasks which might surface such reasoning lapses for feedback to improve learning.

Background

Since the National Council of Teachers of Mathematics [NCTM] suggested that "problem solving be the focus of school mathematics in the 1980s" in their publication the *Agenda for Action* (NCTM, 1980, p. 1), there has been a widespread realignment of mathematics pedagogy to incorporate mathematical problem solving into the curriculum. Singapore is no exception. In 1990, the Singapore Ministry of Education [MOE] embraced problem solving as the primary focus of its mathematics curriculum. Even though the curriculum went through a few revisions, the latest being in 2006, problem solving has remained the focus of the Singapore mathematics curriculum. Along with the latest revision to the Singapore mathematics curriculum, new assessment objectives were developed and included. There are now three levels of assessment objectives (AO1, AO2, AO3) for the examination of student achievement in mathematics at the Singapore-Cambridge GCE

A-Level examination[1] (SEAB, 2007). It is envisioned that the assessment will test the students' abilities to:

AO1 understand and apply mathematical concepts and skills in a variety of contexts, including the manipulation of mathematical expressions and use of graphic calculators;

AO2 reason and communicate mathematically through writing mathematical explanation, arguments and proofs, and inferences;

AO3 solve unfamiliar problems; translate common realistic contexts into mathematics; interpret and evaluate mathematical results, and use the results to make predictions, or comment on the context.

These objectives require that the assessment make informed judgements of a student's conceptual understanding and mathematical reasoning process. However as we have briefly discussed in the introduction, it is hard to make valid inferences about the mental processes of the student simply by the things one writes on a piece of paper (Skemp, 1978). This presents the difficulty of providing sound feedback, which is fundamental to formative assessments, in a traditional pen and paper mathematics assessment.

Methodology

This study is part of a larger qualitative study that was carried out by the author in a Junior College[2] [JC] in Singapore. The participants in the study were 69 second year JC students[3] of mixed abilities from four different classes. Among the 69 students, 28 of them were females and 41 were males. These students had finished the JC curriculum and were preparing for their Preliminary Examinations[4] which is often treated as a benchmark to students' preparation for the nationally administered Singapore-Cambridge GCE A-Level examination. A written test which included 16 free response questions, covering a range of topics from the A-Level Pure Mathematics syllabus, was constructed and administered to the participants. The items in the test were piloted with a small group of 6 students, not from the sample, and as a result some of the items were refined. Since the items were constructed to sieve out students with errors in mathematical reasoning, the items demanded knowledge of fine details to the mathematical concepts so that students who had applied procedures without reasons might be flagged out. As the objective was to investigate students' mathematical reasoning processes and not test their ability to manipulate expressions, the expressions were consciously kept simple so that students would not be intimidated by complicated looking expressions or be hindered by tedious manipulations. So while the items were not the common ones encountered by students in their tutorial assignments, they were within the students' capabilities. The students were told to take the test as an indication of their preparation for the preliminary examinations. As such the motivation of the students was high. The test was marked and the participants' errors were documented. As an indication of how common the errors were, the proportion of the participants making similar errors was computed. A semi-structured interview was conducted with selected participants to probe into the reasoning of the students. Students from across the range of results were represented in the interview. Selected students took the interview as a feedback session, and so were

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[1] The Singapore-Cambridge GCE A-Level examination is a national public examination at the end of Grade 12. Its result is used for university admissions.

[2] Junior College is a two year programme equivalent to Grade 11 and Grade 12.

[3] Typical age of Grade 12 students are 18 years old.

[4] The Preliminary Examinations is an internal school examination at the end of the JC course in which students are tested on what they are taught over the two years.

forthcoming with their responses. The students who had the correct answers were also asked to comment on other students' presentations.

Throughout this study, the male-centric pronouns, e.g., "he" will be used to represent both the male and female gender and "I" is used to denote myself, the interviewer. All names used are pseudonyms.

Results and Discussion

A key finding in this study is that generally students do mathematics by recalling and executing procedures, often applying rules without reasons. In Skemp's (1978) terms, these students possessed an *instrumental understanding* of the concepts. This is opposed to those who possess *relational understanding* where they know the purpose and reasons behind the procedures and why they work. Skemp had confessed that initially, he did not regard instrumental understanding as understanding at all. However, Skemp realised that many students as well as teachers do regard the possession and the ability to use the rules as understanding. Skemp acknowledged that in a restricted meaning of the word, these students do understand. However as we shall see, even though such students may be able to apply rules correctly, they may not be able to justify their arguments mathematically.

The possession of instrumental understanding is evident in a number of students' answer to the item on mathematical induction, which is Question 2 in the test instrument (see Figure 1). The procedural nature of students' thinking is exemplified in the following interview excerpt with Ted who had done the problem correctly:

I: Can you explain to me the proof by mathematical induction?
Ted: First you must let $p(n)$ be …write down the whole question. Then you prove $p(1)$ true, then you assume $p(k)$ true for something. Then you consider the $p(k + 1)$, then must try and get ...say if I put $(k + 1)$ into the right hand side. Then if it is true then all true.

Such a response is not unique to Singapore students. Similar answers depicting the procedural nature of students' thinking had been documented by Baker (1996) in his study on the difficulties faced by students in mathematical induction. It is not uncommon that students memorise rules and procedures without understanding them, some of whom incessantly master the skills and algorithms before even trying to gain an appreciation into the "why" behind those procedures. Skemp (1978) noted that some teachers may also make a reasoned choice to teach instrumentally as its rewards can be more immediate and more apparent. For some students, relational understanding does follow from these rules learned instrumentally, albeit possibly after some reflection, but for many, relational understanding may be practically out of reach (Linchevski & Sfard, 1991). When students merely follow formulas instrumentally, they may think that rote learning rather than creativity and discovery is the key to successful problem solving. Such students are unlikely to gain confidence in their ability to create mathematics and will resort to other means to justify their mathematical arguments (Harel & Sowder, 1998). This is evident in the continuation of the interview with Ted:

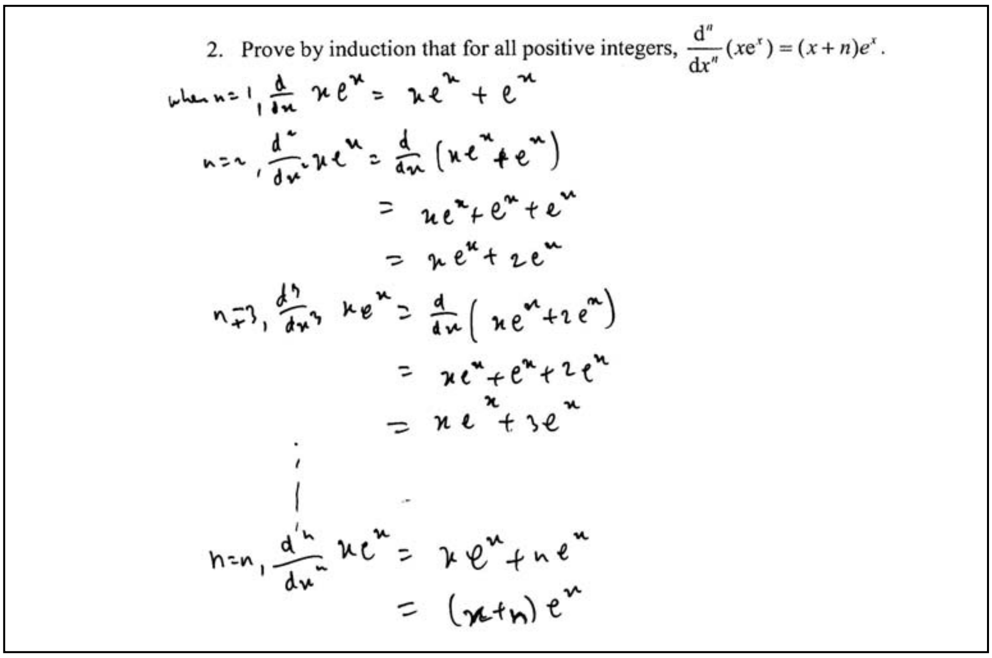
I: It (mathematical induction) is a proof. Why does this process work?
Ted: I don't know.
I: You don't know?
Ted: They tell me to do like that, so I do like that. This is the format for induction.

So in addition to Ted's procedural description of mathematical induction, the interview also revealed his appeal to convictions outside of mathematical reasoning to justify his mathematical arguments.

External Conviction Proof Schemes

Ted's justification of the proof by mathematical induction was based on "they tell me to do like that, so I do like that" and by appealing to the "format of induction". Explanation and justification are key aspects of students' mathematical activity in classrooms where mathematics is reasoning (Yackel & Hanna, 2003). However, in the absence of relational understanding, Ted resorted to justifying his arguments by appealing to a higher authority (authoritarian proof scheme) and judged the correctness of the proof based on the form of the argument (ritual proof scheme) (see Sowder & Harel, 1998). Such learning habits are the result of instrumental learning (Harel & Sowder, 1998). A dash of the authoritarian proof scheme is perhaps unavoidable and not completely detrimental. It, however, becomes a problem if the student relies totally on the authority without question. In its worst forms, the student either regards the justification of a mathematical argument as worthless and unnecessary, or is helpless without an authority at hand (Harel & Sowder, 1998). In such cases, it is not uncommon that whenever such students encounter difficulties, they ask for help without first making a serious effort to solve it on their own.

The pitfall of the ritual proof scheme is that a student will mistake the form for the mathematics. The continuation of my interview with Ted illustrates how he was paralysed by the format of mathematical induction and was therefore not able to explain why another presentation (Figure 1) was not acceptable.


2. Prove by induction that for all positive integers, $\frac{d^n}{dx^n}(xe^x) = (x+n)e^x$.

when n=1, $\frac{d}{dx} xe^x = xe^x + e^x$

n=2, $\frac{d^2}{dx^2} xe^x = \frac{d}{dx}(xe^x+e^x)$
$= xe^x+e^x+e^x$
$= xe^x+2e^x$

n=3, $\frac{d^3}{dx^3} xe^x = \frac{d}{dx}(xe^x+2e^x)$
$= xe^x+e^x+2e^x$
$= xe^x+3e^x$

⋮

n=n, $\frac{d^n}{dx^n} xe^x = xe^x+ne^x$
$= (x+n)e^x$

Figure 1 Inductive proof scheme

I: In that case, why don't you do this? (pointing to Figure 1)
Ted: Because this is not induction.
I: Why is this not induction?
Ted: Induction is that one (pointing to his own correctly presented solution).
I: Why is induction that one and not this (pointing to Figure 1)?
Ted: Because we learned that one.

Inductive Proof Scheme

The problem with instrumental understanding and its consequences is that in most cases, they cannot be detected through students' answers on pen and paper, especially so when students provide textbook answers to routine problems. So while a page of correctly presented piece of work is a good indicator that a student has learned the procedure of solving the problem, it does not tell if the student has understood the mathematical concepts or if he has acquired the reasoning abilities to solve the problem. This means that the work submitted by students who have derived answers instrumentally will not be able to function as a reliable source of information for feedback. Ted is one such example. He had presented his proof by mathematical induction correctly, but when asked why he could not have presented it inductively as in Figure 1, Ted said that the inductively presented answer was wrong solely because that was not in the format of mathematical induction.

I: If you are the teacher, will you mark it correct or wrong?
Ted: Wrong.
I: Why?
Ted: I mean it is correct, but it is not induction.

Another student Kim, whose presentation of the proof was largely correct, had the same perception as Ted and would accept a proof presented inductively as in Figure 1, but not in this context just because of the word "induction".

Kim: It (Figure 1) is not acceptable because the question wants you to prove it by induction.
I: So what if I just ask you to prove – take away the word induction?
Kim: Then it will be correct.

When I asked John, who had stumbled in the induction step of his proof, if Figure 1 was a valid proof, he replied:

John: Hmm… It is logical. I think if I can think of it, I will also do this.

Mathematical reasoning and everyday reasoning are essentially very different. Mathematics is a deductive science. The communication of mathematical arguments in mathematics is structured such that careful attention must be given to logical deduction. Reasoning in our everyday life does not entail the demands of the rigor required in a formal mathematical proof. As Polya (1941) put it, "rigorous, precise, properly so-called logical reasoning is found in its pure form only in mathematics" (p. 450). Although in this study, only two students presented their arguments inductively as in Figure 1, the interviews have uncovered others who would accept an inductive argument as a proof. These students possess the inductive proof scheme (Harel & Sowder, 1998). Such students have limited understanding of mathematical proofs and do not appreciate that a mathematical proof must be rigorous, general, complete and conclusive.

In the case of Ted, Kim and John, the possession of the inductive proof scheme would have gone unnoticed apart from the interview. They are examples of students' with sorely inadequate knowledge, but instrumental understanding had masked their incompetence and precluded necessary feedback. The assessment results obtained in such cases would just be an indicator of task completion; they provide little information on the students' conceptual understanding.

The continuation of the interview with John revealed that he accepted the solution of Figure 1 as correct because of his prior experience with questions that required him to deduce the answer from patterns.

I: You also will do this (pointing to Figure 1)?

John: Why not? Because in secondary school, it is also like that. There is always .... this kind of question is always the last part. If you haven't learned how to do mathematical induction, then ….. this will be …ya…its very possible.

I: You think you can proof something by the recognition of patterns?

John: Ya….patterns …..because Maths is about patterns.….

In elementary mathematics, students are encouraged to explore, often without the need for rigour. An example would be the pattern explorations they do in secondary schools. At that level, once students are able to come out with the general mathematical statement from the observed patterns, the problem is considered solved. John did not appreciate that the demand in a formal proof is much greater than deriving a conjecture. To him, they were the same "because in secondary school, it is also like that" and "because Maths is about patterns". The shift from elementary to more advanced mathematics is never a smooth transition, and assessment tasks may either aid or hinder the progress.

Citing Dumas-Carre & Larcher (1987), Black and William (1998) remarked that there are various kinds of assessment tasks: those that are identical to the ones studied, those that are typical but not identical, and those that present a new problem requiring the construction of new approaches and the deployment of established knowledge in new ways. If assessments consistently consist of tasks which are identical to the ones taught, then students will develop a narrow conception of mathematics problems.

Harel and Sowder (1998) observed that the kinds of problems typically introduced to students in their first experience with mathematical induction have been cognitively inadequate. Ernest (1984) noted that the frequent use of examples involving finite series in mathematical induction has led to a common misconception concerning the form of the $p(k+1)$ statement in the proof. In such cases, students simply assume that the $p(k+1)$ statement equals "the expression given in the $p(k)$ statement" + "something", as in the solution in Figure 2. Students derive this habit from mindlessly doing many mathematical induction questions where they begin their proof of the $p(k+1)$ statement with $\sum_{r=1}^{k+1} u_r = \sum_{r=1}^{k} u_r + u_{k+1}$.

Assuming P_k true ie " $\frac{d^k}{dx^k}(xe^x) = (x+k)e^x$ "

Testing P_{k+1} ie to prove true " $\frac{d^{k+1}}{dx^{k+1}}(xe^x) = (x+k+1)e^x$ "

$$\text{LHS} = \frac{d^{k+1}}{d^{k+1}}(xe^x)$$

$$= (x+k)e^x + \frac{d}{dx}(xe^x)$$

$$= (x+k)e^x + e^x(x+1)$$

Figure 2 Common analogical reasoning errors in mathematical induction

These are typical analogical reasoning errors where students simply apply a procedure from one context to a different context without giving much thought to the deeper structural properties of the mathematical problem in each context.

Analogical Reasoning Errors

Inference by analogy is one of the most essential and widely used problem solving strategies (Polya, 1957). Analogical reasoning entails understanding something new by comparing with something that is known (English, 1998). It is generally defined as the transfer of structural information from one system (called the source), to another system (called the target) through mapping relational correspondences between the two systems (English & Sharry, 1996). This transfer is achieved through identifying common problem structures between the two systems and applying known procedures from one to the other. This requires a relational understanding of both the source and target systems. Instrumental understanding, on the other hand, limits students' transfer of what they have learned to a new environment. One of the errors novice problem solvers make is that they focus on the superficial features rather than on the underlying relational structural properties between the source and target problems (English, 1998). This is clear from the responses to Question 3 of the test instrument which consists of two items on inequalities (see Figure 4). The inequality items were constructed such that the answer to one of them had a singular value, not like typical ones where the answers are usually a range of values. The other inequality item is unlike familiar ones in which the given expression may be simplified into linear factors.

The most glaring errors were inequality statements that did not make sense. Figure 3 shows two students' solutions where they drew number lines and formed inequalities which included complex numbers.

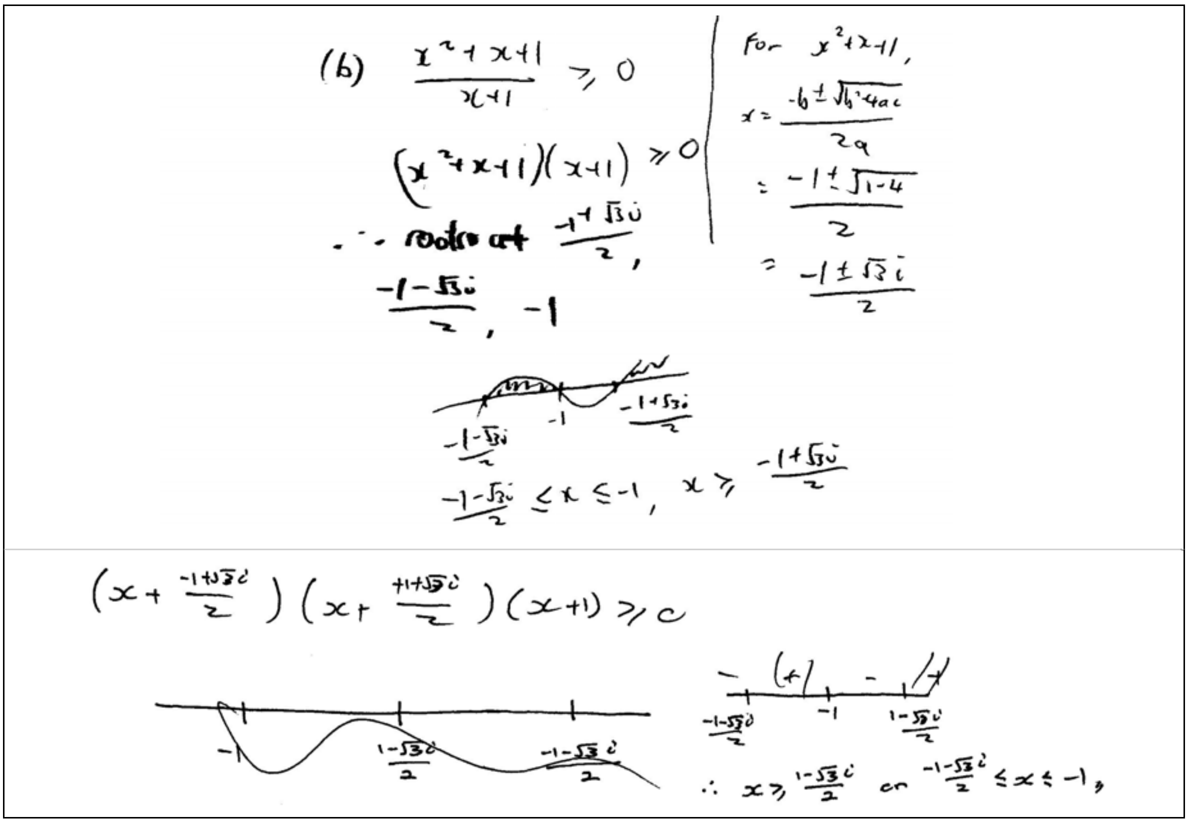


Figure 3 Inequalities with complex numbers

Typically, such errors arise because students treat inequalities like equations and they draw inappropriate analogies between the solution structures of equations and inequalities (Tsamir, Almog & Tirosh, 1998). Equations and inequalities are structurally very different: "=" is an equivalence relation while "≤" is not. There are many procedures which can be applied to equations but not to inequalities because in so doing the truth value of the inequality statement will not be preserved. However, students simply assume that the same solution procedure holds for both. Two examples are shown in Figure 4.

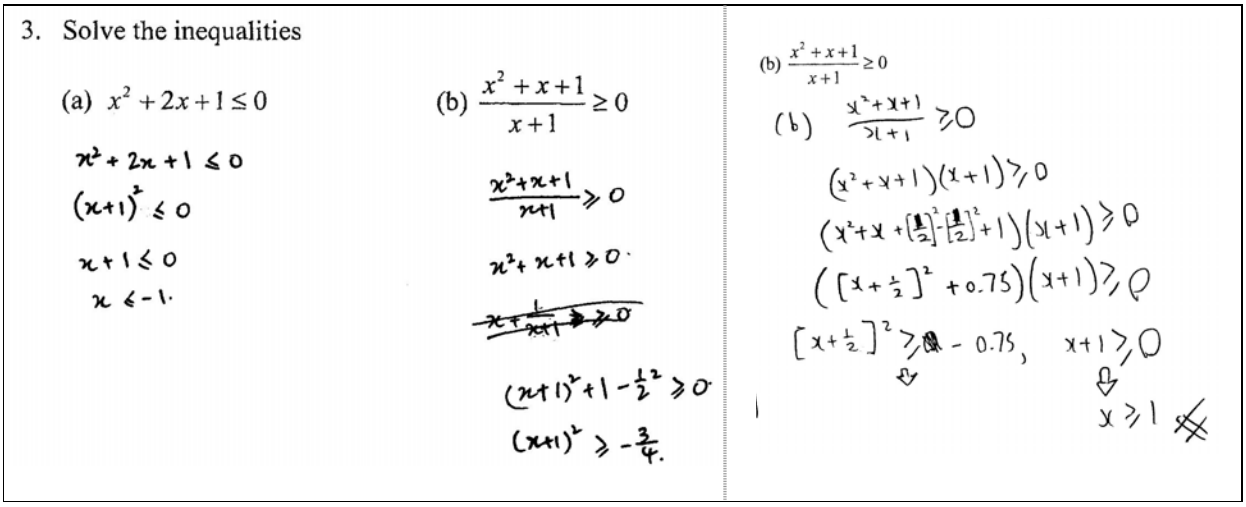
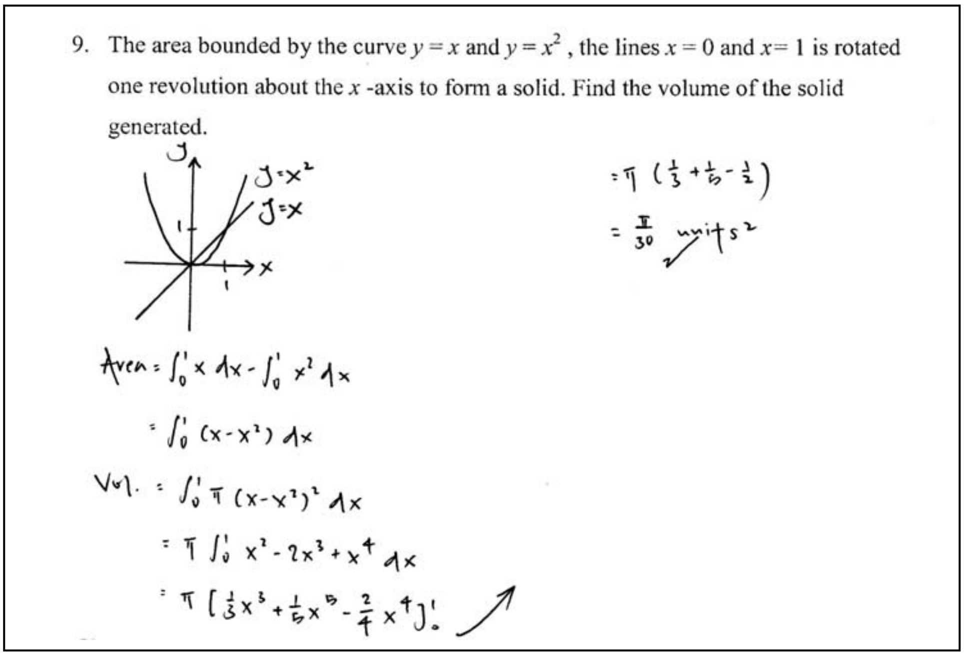


Figure 4 Solving inequalities like equations

These students had failed to recognize that the properties underlying valid equation-solving transformations are not the same as those underlying valid inequality-solving transformations (Kieran, 2004). During the interview, Kim admitted that his errors originated from a general ignorance of the structural differences between equations and inequalities:

Kim: I was trying to treat the inequality like an equation…… which I know is not thoroughly correct.

The treatment of inequalities like equations also highlighted an underlying problem: that in their manipulation of algebraic expression, students did not consider if the procedures preserve the truth value of the mathematical statements. Cortes and Pfaff (2000) noted that some students, when working with equations, do not provide any mathematical justification for transformations on the equations. Instrumental understanding acquired at earlier stages can influence the learning of new concepts introduced at a later stage. Such interference is also evident in Question 9 (see Figure 5) of the test instrument where students had to find the volume of the solid generated by a region enclosed between two curves. This item requires a student to recognise that unlike solids which are generated by area bounded by a curve and the *x*-axis, the solid generated from the context given has a hollow centre. In the study, 18.8% of the participants made the error of using $\int_0^1 \pi(x-x^2)^2\,dx$ to find the volume. This accounted for almost half of the errors made in the sample. Students who produced this solution did not consider how the hollow centre of the resultant solid would have affected the mathematics. Figure 5 shows the question and an example of the error.


9. The area bounded by the curve $y = x$ and $y = x^2$, the lines $x = 0$ and $x = 1$ is rotated one revolution about the x-axis to form a solid. Find the volume of the solid generated.

Area $= \int_0^1 x\,dx - \int_0^1 x^2\,dx$

$= \int_0^1 (x - x^2)\,dx$

Vol. $= \int_0^1 \pi (x - x^2)^2\,dx$

$= \pi \int_0^1 x^2 - 2x^3 + x^4\,dx$

$= \pi \left[\frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{2}{4}x^4\right]_0^1$

$= \pi\left(\frac{1}{3} + \frac{1}{5} - \frac{1}{2}\right)$

$= \frac{\pi}{30}$ units^2

Figure 5 Analogical reasoning error

Chen's response in the interview shed some light on the reasoning behind the error.

I: What gave you that idea? Why did you think like that?
Chen: Because of the area. Because area you can minus away.

Chen's reply indicated that these students had erroneously linked the formula of finding the volume generated by the region bounded by the two curves to that of the area of a bounded region. Singapore students learn how to use integration to find the area of a bounded region in Additional Mathematics in secondary 4 (Grade 10) and proceed on to learn how to calculate the volume of a solid generated by rotating a bounded area about the x or y axis in Junior College. However, many students simply memorise $\int_a^b [f(x) - g(x)]\,dx$ as a formula to use for finding area bounded by two curves and erroneously extend it to $\pi\int_a^b [f(x) - g(x)]^2\,dx$ to find the volume generated by the area bounded. They do not realise that the former formula is derived from the property that $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$. This demonstrates that common analogical reasoning errors occur because of discontinuities in the acquisition of mathematical knowledge. Assessment tasks which target at such discontinuities may be able to gather the necessary information to bridge the discontinuities. The exploration of students' prior knowledge through their answers in new contexts improves learning because it helps students relate the new to the old and helps avoid superficial treatments of the new (Black & William, 1998).

Implication of Findings for Assessment

The above discussion highlighted two reasoning lapses uncovered through the use of novel items: first, students had mistakenly accepted inductive strategies as valid mathematical proofs; second, students made analogical reasoning errors in their transfer of procedures from source systems to target systems. The students' over-reliance on instrumental understanding had a large part to play in these reasoning lapses. Procedures that were understood instrumentally were erroneously applied from previously acquired knowledge into new environments. The lack of relational understanding presents a difficulty in gathering accurate information about students' learning. Reasoning lapses could be deceitfully masked by instrumental understanding making them difficult to elicit through familiar questions. Thus for teachers to be able to get evidence for appropriate follow up actions, traditional assessments may not be adequate. Instead, conscious effort must be made to construct items in the light of specific knowledge structures so that teachers may determine the progress of the students to help them overcome their discontinuities in learning. The construction of these items requires teachers to predict what students can learn as much as it requires them to have knowledge in what their students have already learned (Black & William, 1998). Such items could spark off the assessment cycle of elicitation and make possible assessment for learning in the classroom.

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